A COMPARATIVE STUDY OF THE BLACK-SCHOLE MODEL AND A MODIFIED BLACK-SCHOLE MODEL WITH BOUNDED UNDERLYING PRICES FOR OPTION VALUATION

A B M Shahadat Hossain and Farzana Afroz
Department of Applied Mathematics, University of Dhaka, Dhaka-1000, Bangladesh
Corresponding Author: abmsh@du.ac.bd, farzana.afrz123@gmail.com

Abstract

In this paper, a modified Black-Scholes model is studied which assumes that the price of the underlying asset varies within a finite zone rather than being allowed to vary in a semi-infinite zone as considered in the Black-Scholes model. At first, the arbitrage free condition is imposed to get the closed form pricing formula of the modified Black-Scholes model for European put options. Then a number of basic properties of the solution such as, monotonicity, affecting factors of the Black-Scholes model are tested for the modified Black-Scholes model. Moreover, we present the condition when two model coincides both theoretically and numerically. Finally, we derive the formulas of Delta and Gamma for the modified Black-Scholes model and investigate their correctness by comparing them with Black-Scholes’s Delta and Gamma.

Keywords Truncated Normal Distribution, Martingale Approach, Delta, Gamma

1 Introduction

Derivatives have become essential in finance and derivative like options are actively traded on many exchanges throughout the world. Accurate option pricing is a challenging task. So appropriate option pricing model is always desirable in financial market. The Black-Scholes (BS) model was developed by Fisher Black, Robert Merton and Myron Scholes in 1973 [12], which is regarded as one of the best ways of determining fair prices of options. But sometimes the B-S model gives captious results due to the fundamental assumptions made in this model. For example, observed returns of the underlying asset from financial markets are actually not normally distributed as the B-S model considered. Moreover, B-S model assumes that the underlying asset price varies within a semi-infinite zone. But in reality, underlying asset price can never reach infinity. As well as, the model believe that the volatility term will be constant which is not actually possible in real markets. Considering this facts of the B-S model, several modification of this model have been introduced to get more accurate option prices.

The constant volatility assumption is mitigated by several proposed option pricing model like stochastic volatility model by Heston [9]. Another modification type is replacing Brownian motion with other stochastic processes. For example, jump-diffusion model [10] shows that
the underlying price is discontinuous in real market by adding a jump term to the standard Brownian motion.

An alternative type of modification is presented by only specifying the probability density function of the underlying asset price and imposing the martingale approach. Suitable distribution can describe different characteristics of the asset returns and volatility term structures which are not clearly explained by the B-S model. In this paper, we are interested to analysis a modified Black-Scholes (B-S) model which is based on distributional modification, proposed by Xin-Jiang He & SongPing Zhu in 2018 [1]. The model adopt truncated normal distribution to develop option pricing formula. As the truncated normal distribution is chosen in option pricing, there would be a price range for the underlying asset during a certain period. The price range will be depended on traders own expected range. So that, if a trader believes that the price of the asset will not exceed a certain level, he/ she can greatly be benefited by choosing this modified B-S option price.

Here, we investigate the modified B-S model for different fundamental properties of the B-S model and extend it’s properties. We derive Delta and Gamma of the modified B-S model which can be used by traders to observe option price sensitivity in financial markets. In our work, we consider European put option for the modified B-S model while Xin-Jiang He & Song-Ping Zhu developed this model for European call option [1].

The paper is structured as follows. In Section 2, we present the pricing formula of the B-S model and then we briefly discuss about truncated normal distribution. After that, we derive necessary martingale restriction for the modified B-S model and develop its closed form pricing formula for European put options. In section 3, we provide some important results for the modified B-S model. Numerical experiments and comparisons are showed in section 4. Then in section 5, we present some concluding remarks.

2 Our Models

2.1 The B–S Model

The B-S model considers that the dynamics of a risky asset, such as stock, follows Stochastic Differential equation (SDE):

\[ dS(t) = rS(t)dt + \sigma S(t)dB(t), \quad \forall t \in [0,T] \]

\[ S(0) = S_0, \]

where \( B(t) \) is a Brownian motion under the given probability measure \( Q \), called the probability measure of the risk-neutral world.

2.1.1 B–S European Put Price Formula

The value of a European put option at time \( t \) with exercise price \( K \), current stock price \( S_0 \), maturity \( T \), constant interest-rate \( r \) and volatility \( \sigma \):

\[ V_p^{BS} = Ke^{r(T-t)} \Phi(-d_2) - S_0 \Phi(-d_1) \]

(2)

where

\[ d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = d_1 - \sigma \sqrt{T-t}. \]

Here \( \Phi(z) \) is the standard normal cumulative distribution function, given by

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx, \]

(3)

\( \Phi(x) \) is the probability that a variable with a standard normal distribution with mean 0 and variance 1, i.e. \( N(0,1) \), will be less than \( x \).
2.2 The Modified B–S Model

On the other hand, the pricing formula of the modified B-S model is obtained through truncated normal distribution and martingale approach.

2.2.1 Some Notations

We setup the following notations in the proceeding descriptions of the paper

\[ \eta_a = \frac{a - \mu T}{\sigma \sqrt{T}}, \quad \eta_y = \frac{\ln y - \mu T}{\sigma \sqrt{T}}, \quad \eta_b = \frac{b - \mu T}{\sigma \sqrt{T}}, \quad \eta^*_a = \frac{\ln(S_0) - \mu T}{\sigma \sqrt{T}}, \quad \eta^*_y = \frac{\ln y - \mu T}{\sigma \sqrt{T}} - \sigma \sqrt{T} \]

where, \( a, b \) denotes lower and upper truncation points of truncated normal distribution.

2.2.2 Truncated Normal Distribution

In Fig.1, we see that for two different pairs of truncation points, truncated normal distribution holds higher probability in the truncated area than the standard normal distribution. Also for different choices of \( a, b \), the probability can be narrowed down. \[5\]

Now the modified B-S model considers that the underlying log price follows a truncated normal distribution under the martingale measure \( Q \),

\[ \ln \left( \frac{S_T}{S_0} \right) \sim f_r(x; \mu T, \sigma \sqrt{T}, a, b) \]

(4)

which means that, the underlying price \( S_0 \) will be higher than \( S_0 e^a \) but lower than \( S_0 e^b \).

We know, the goal of perfect hedging at any time \( t \in [0, T] \) can be achieved under the martingale framework, which implies that the modified B-S model will be arbitrage-free if we impose the condition \[2\],

\[ E^Q\left[ e^{-rT} S_T \right] = e^{-rT} S_0 \]

(5)

As the condition in (5) is applied in B-S model, the drift \( \mu \) could be shifted to \( r - \frac{1}{2} \sigma^2 \) that means a reduction in the parameter space \[8\]. So this condition is applied in the new model too, this implies that the expected return \( \mu \) must be a function of \( r \) and \( \sigma \) as well as two bounds \( a \) and \( b \) \[1\]. If we denote \( Y = \frac{S_T}{S_0} \), we can easily find the probability density function for \( Y \), can be expressed as \[6\], leading to

\[ f_{Y_T}(y) = \begin{cases} \frac{1}{y} \frac{1}{\sqrt{2\pi} \sigma T} \phi(\eta_y), & e^a \leq y \leq e^b \\ 0, & \text{otherwise} \end{cases} \]

(6)

before imposing the arbitrage-free condition, we need to consider the following proposition,
Proposition 1. The conditional expectation of $Y = \left( \frac{S_T}{S_0} \right)$ given the information up to time $t = 0$ under truncated normal distribution is

$$E[Y|\mathcal{F}_0] = e^{\mu T + \frac{\sigma^2 T}{2}} \frac{\Phi(\eta_0) - \Phi(\eta_a)}{\Phi(\eta_b) - \Phi(\eta_a)}$$  \hspace{1cm} (7)

Proof. We know,

$$E[Y|\mathcal{F}_0] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} \phi(\eta_0) - \phi(\eta_a)} dy.$$  \hspace{1cm} (8)

Now letting $z = \frac{\ln y - \mu T}{\sigma \sqrt{T}}$ we get,

$$E[Y|\mathcal{F}_0] = \int_{-\infty}^{\infty} e^{\mu T + \frac{\sigma^2 z}{2}} \frac{\Phi(\eta_0 - \sigma \sqrt{T}) - \Phi(\eta_b - \sigma \sqrt{T})}{\Phi(\eta_b) - \Phi(\eta_a)} dz.$$  \hspace{1cm} (9)

So from equation (5), we get,

$$E[e^{-\mu T} S_T|\mathcal{F}_0] = S_0$$

$$\Rightarrow E[Y|\mathcal{F}_0] = e^{\mu T}.$$  \hspace{1cm} (10)

Therefore following can be obtained using the above proposition and (9).

$$\Phi(\eta_0) - \Phi(\eta_a) = e^{(\mu - \frac{1}{2}\sigma^2 - \mu) T}.$$  \hspace{1cm} (11)

Here, we see that, $\mu$ is an implicit function of given parameters and time to maturity for the underlying options. Which means that if $a, b, \sigma, r, T$ are given $\mu$ can be calculated from equation (10) as a “root finding” problem. After the martingale restriction is imposed, we are now ready to derive a closed-form pricing formula for European options under the modified B-S model with martingale approach, which will be provided in the next subsection.

2.2.3 Modified B–S Pricing Formula

The pricing formula for European type of options under the modified B-S model is obtained considering three cases with respect to the initial underlying price $S_0$ and the strike price $K$ of the option. The first one is when

$$\frac{K}{S_0} < e^\mu,$$

we know $S_T \geq K$ always holds, which tells us that the put option is worthless. On the other hand, when

$$\frac{K}{S_0} > e^\mu,$$

then $K \geq S_T$ always holds. So the put option price can be easily obtain as $Ke^{-rT} - S_0$.

Now we consider the third case, in the following proposition to obtain the pricing formula of the put option using the definition of martingale restriction.

Proposition 2. The price of a European put option for the modified B-S model with strike $K$ and maturity $T$ is given by

$$V_p^{\text{B-S}} = Ke^{-rT} \frac{\Phi(\eta_a) - \Phi(\eta_b)}{\Phi(\eta_a) - \Phi(\eta_b)}.$$  \hspace{1cm} (12)

Proof. We know the value of the option is

$$V_p^{\text{B-S}} = e^{-rT} E[\max(K - S_T, 0)|\mathcal{F}_0]$$

$$= S_0 e^{-rT} \int_{-\infty}^{\infty} \max\left( \frac{K}{S_0} - y, 0 \right) f_Y(y) dy$$

$$= \frac{S_0 e^{-rT}}{\Phi(\frac{\ln y - \mu T}{\sigma \sqrt{T}}) - \Phi(\frac{\ln y - \mu T}{\sigma \sqrt{T}})} \left[ \int_{-\infty}^{\frac{\ln y - \mu T}{\sigma \sqrt{T}}} \frac{K}{S_0} \Phi \left( \frac{\ln y - \mu T}{\sigma \sqrt{T}} \right) dy \right. - \left. \int_{\frac{\ln y - \mu T}{\sigma \sqrt{T}}}^{\infty} \frac{1}{\sigma \sqrt{T}} \Phi \left( \frac{\ln y - \mu T}{\sigma \sqrt{T}} \right) dy \right].$$
after simplifying \( A_1 \) and \( A_2 \) we get,

\[
A_1 = \frac{K}{S_0} [\Phi(\eta_b) - \Phi(\eta_a)]
\]

\[
A_2 = -e^{(\mu+\frac{1}{2}\sigma^2)T} \left[ \Phi(\eta^-_b) - \Phi(\eta^-_a) \right]
\]

substituting the simplified values of \( A_1 \) and \( A_2 \) into eqn (12)

\[
V_{p,MB} = \frac{S_0 e^{-rT}}{\Phi(\eta_b) - \Phi(\eta_a)} \left[ \frac{K}{S_0} (\Phi(\eta_b) - \Phi(\eta_a)) - e^{(\mu+\frac{1}{2}\sigma^2)T} \left[ \Phi(\eta^-_b) - \Phi(\eta^-_a) \right] \right]
\]

\[
= K e^{-rT} \left( \frac{\Phi(\eta_b) - \Phi(\eta_a)}{\Phi(\eta_b) - \Phi(\eta_a)} - S_0 e^{\mu T + \frac{1}{2} \sigma^2 T - rT} \frac{\Phi(\eta^-_b) - \Phi(\eta^-_a)}{\Phi(\eta^-_b) - \Phi(\eta^-_a)} \right)
\]

Now using eqn (10) we obtain the required put price formula,

\[
V_{p,MB} = K e^{-rT} \left( \frac{\Phi(\eta_b) - \Phi(\eta_a)}{\Phi(\eta_b) - \Phi(\eta_a)} - S_0 \frac{\Phi(\eta^-_b) - \Phi(\eta^-_a)}{\Phi(\eta^-_b) - \Phi(\eta^-_a)} \right)
\]

\[
\frac{\partial V_{p,MB}}{\partial S} = \frac{K e^{-rT}}{S_0 \sigma \sqrt{T}} \left( \frac{\phi(\eta_b) - \Phi(\eta_b)}{\Phi(\eta_b) - \Phi(\eta_a)} - \frac{\phi(\eta^-_b) - \Phi(\eta^-_b)}{\Phi(\eta^-_b) - \Phi(\eta^-_a)} \right)
\]

\[+ \frac{1}{\sigma \sqrt{T}} \left( \frac{\phi(\eta^-_b)}{\Phi(\eta^-_b) - \Phi(\eta^-_a)} \right).\]

Letting,

\[
\frac{\partial V_{p,MB}}{\partial S} = M_1 - M_2 + M_3
\]

\(M_3\) can be simplified as,

Using equation (10) we get,

\[
M_3 = \frac{1}{\sigma \sqrt{T}} \left( \frac{1}{\Phi(\eta^-_b) - \Phi(\eta^-_a)} \right) \left( e^{-\frac{1}{2} (\eta^-_b)^2 + \ln(\frac{K}{S_0}) / \sqrt{2\pi}} \right)
\]

\[
\times e^{-rT} \frac{\phi(\eta^-_b) - \Phi(\eta^-_b)}{\Phi(\eta^-_b) - \Phi(\eta^-_a)}
\]

\[
= \frac{1}{\sigma \sqrt{T}} \left( \frac{1}{\Phi(\eta^-_b) - \Phi(\eta^-_a)} \right) \left( e^{-\frac{1}{2} (\eta^-_b)^2 + \ln(\frac{K}{S_0}) - \frac{1}{2} \sigma^2 T} \right)
\]

\[
= \frac{K e^{-rT}}{S_0 \sigma \sqrt{T}} \frac{\phi(\eta^-_b)}{\Phi(\eta^-_b) - \Phi(\eta^-_a)}
\]

\[
= M_1
\]

(14)

Now using eqn (10) we obtain the required put price formula,

\[
\frac{\partial V_{p,MB}}{\partial S} = -M_2.
\]

Moreover, we know that the normal distribution function \( \Phi(x) \) is a monotonic increasing function of \( x \). In addition, with \( e^x > \frac{K}{S_0} > e^\mu \), we can also obtain that,

\[
\eta^-_b \geq \eta^-_b \geq \eta^-_a
\]

which implies that \( M_2 > 0 \). Therefore, we have found that \( \frac{\partial V_{p,MB}}{\partial S} < 0 \). This completes the proof.

3 Some Results for the Modified B-S Model

**Proposition 3.** The European put option price \( V_{p,MB} \) is a monotonic decreasing function of the underlying price \( S \).

**Proof.** To show the monotonicity of the option price with respect to \( S \), we have,
So the modified B-S pricing formula is correlative with this fundamental property of the put price formula.

**Proposition 4.** The put option price $V_p^{\text{MBS}}$ degenerates to the B-S put price $V_p^{\text{BS}}$ as $a$ and $b$ approach $-\infty$ and $+\infty$ respectively.

**Proof.** To verify, we first need to do the following calculations. We know that,

$$\Phi(\eta_b) = \Phi\left(\eta_b - \sqrt{T}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta_b - \sqrt{T}} e^{-\frac{x^2}{2}} \, dx.$$

As we consider that,

$$\lim_{b \to +\infty} \Phi(\eta_b) \approx \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}} \, dx = 1$$

Similarly we can find,

$$\lim_{b \to +\infty} \Phi(\eta_b) = 1, \quad \lim_{a \to +\infty} \Phi(\eta_a) = 0, \quad \lim_{a \to -\infty} \Phi(\eta_a) = 0.$$

As a result, the martingale restriction (10) under the limitation of $a$ and $b$ can be simplified as,

$$\mu = r - \frac{1}{2} \sigma^2$$

which is exactly the same as the one for the B-S model. Moreover, the following can be obtained as,

$$\lim_{a \to +\infty} \lim_{b \to +\infty} V_p^{\text{MBS}} = Ke^{-rT} \left\{ -\Phi\left(\eta_a\right) + \Phi\left(\eta_b\right) \right\} - S_0 \Phi\left(\eta_b\right) + \Phi\left(\eta_a\right)$$

$$= Ke^{-rT} \Phi\left(\eta_a\right) - S_0 \Phi\left(\eta_b\right)$$

using equation (16)

$$\lim_{a \to +\infty} V_p^{\text{MBS}} = Ke^{-rT} \Phi\left(\eta_a\right)$$

which is exactly the B-S put price. This completes the proof.

### 3.1 Delta and Gamma for the Modified B–S Model

In this section, we derive option’s risk sensitivity Delta and Gamma for the modified B-S put option formula $V_p^{\text{MBS}}$.

**Proposition 5.** The Delta of the European put option price $V_p^{\text{MBS}}$ at time $(T - t)$ is given by

$$\Delta_p = -\frac{\Phi(\eta_1) - \Phi(\eta_{\infty})}{\Phi(\eta_b) - \Phi(\eta_a)}$$

**Proof.** We know the Delta of an option is defined as,

$$\Delta_p = \frac{\partial V_p^{\text{MBS}}}{\partial S}$$

$$= -\frac{Ke^{-r(T-t)}}{S_0 \sigma \sqrt{T-t}} \left\{ \Phi(\eta_1) - \Phi(\eta_a) \right\} + \frac{1}{\sigma \sqrt{T-t}} \left\{ \Phi(\eta_b) - \Phi(\eta_{\infty}) \right\}$$

using equation (10), we get
Proposition 6. The Gamma of the European put option price $V_p^{MBS}$ at time $(T-t)$ is given by

$$\Delta_p^{\text{MBS}} = -\frac{Ke^{-(T-t)}}{S_t \sigma \sqrt{T-t}} \left( \phi(\eta_{t}) - \phi(\eta_{t-}) \right)$$

$$+ \frac{Ke^{-(T-t)}}{S_t \sigma \sqrt{T-t}} \left( \phi(\eta_{t+}) - \phi(\eta_{t}) \right)$$

$$\therefore \Delta_p^{\text{MBS}} = -\frac{\phi(\eta_{t+}) - \phi(\eta_{t-})}{\phi(\eta_{t}) - \phi(\eta_{t-})}$$

$$\Gamma_p^{\text{MBS}} = \frac{1}{S_t \sigma \sqrt{T-t}} \frac{\phi(\eta_{t})}{\phi(\eta_{t}) - \phi(\eta_{t-})}$$

4 Numerical Experiments and Comparison

In this section, we make a comparison between the B-S model and the modified B-S model. At first we examine the behavior of option prices for both models varying the upper and lower bound. After that, we show the effect of risk-free interest rate, volatility, stock price, strike price and time to maturity on both models. Then the Delta and Gamma of the modified B-S model for different option positions are compared with the B-S model’s ones. To do all numerical experiments we need to compute $\mu$ using a root-finding scheme. It can be easily done by Matlab built-in function fzero with taking $r - \frac{1}{2}\sigma^2$ as initial guess [1]. We consider the case where, $S_0 = K = 100$, $T = 0.5$, $r = 0.01$, and $\sigma = 0.2$ [1].

We consider more choices of truncation points which are not mentioned in [1] to observe the behavior of modified B-S prices. In both figures, the modified B-S put price coincides to the B-S price at $a = -2$ (see Fig.2(a)) and for $b = 2$ (see Fig.2(b)). We take 10 different pairs of $(a,b)$ in Table 1 to show that our derived modified B-S price coincides to B-S price.

Table 1. Put prices for both models with $S_0 = K = 100$, $T = 0.5$, $r = 0.01$, and $\sigma = 0.2$

<table>
<thead>
<tr>
<th>$(a,b)$</th>
<th>$[S_0 e^a, S_0 e^b]$</th>
<th>$\mu$</th>
<th>$V_p^{\text{MBS}}$</th>
<th>$V_p^{\text{BS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 2. Modified B-S prices vs. B-S prices
Now to observe the impact of affecting factors for both models, we keep fixed \((a,b)\) at two different values and consider the situation where, \(S_0 = 50, K = 50, r = 0.05\) per annum, \(\sigma = 30\%\) per annum, \(T = 1\) [8].

\[
\begin{array}{|c|c|c|c|}
\hline
\text{(ln(0.85),ln(1.15))} & [85,115] & 0.0584 & 20.9208 \\
(-0.15,0.15) & [86.07,116.18] & 0.0111 & 21.2248 \\
(-0.25,0.25) & [77.88,128.40] & -0.0053 & 8.4575 \\
(-0.25,0.35) & [77.88,141.91] & -0.0318 & 7.0784 \\
(-0.35,0.35) & [70.47,141.91] & -0.0090 & 5.8289 & 5.3773 \\
(-0.5,0.5) & [60.65,164.87] & -0.0099 & 5.3887 \\
(-2,0.5) & [13.53,164.87] & -0.00984 & 5.3864 \\
(-0.5,2) & [60.65,738.9] & -0.0102 & 5.3796 \\
(-0.8,0.8) & [44.9,222.5] & -0.0100 & \bf{5.3773} \\
(-2,2) & [13.53,738.9] & -0.0100 & \bf{5.3773} \\
\hline
\end{array}
\]

Now to observe the impact of affecting factors for both models, we keep fixed \((a,b)\) at two different values and consider the situation where, \(S_0 = 50, K = 50, r = 0.05\) per annum, \(\sigma = 30\%\) per annum, \(T = 1\) [8].
We see in Fig. 3, factors that affect B-S prices have almost same impact on modified B-S prices. Moreover, $V_p^{MB5}$ prices are more closer to the B-S prices for $(a,b) = (-0.8, 0.8)$ than $(a,b) = (-0.5, 0.5)$, although the modified prices have similar behavior in both choices of $(a,b)$.

Now we observe how Delta and Gamma behave for both models with respect to the underlying asset. We have already derived them for the Modified B-S model’s in section 3 and Delta and Gamma of the B-S model’s can be found any fundamental financial books [8]. We consider the following example where, $S_0 = 49$, $K = 50$, $r = 0.05$ per annum, $\sigma = 20\%$ per annum, $T = 0.3846(20weeks)$ [8].

![Figure 3. Put prices against affecting factors](image)

**Figure 3.** Put prices against affecting factors

![Figure 4. Delta and Gamma for both models](image)

**Figure 4.** Delta and Gamma for both models with $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.2$, $T = 0.3846$, $(a,b) = (\log(0.85), \log(1.15))$ & $(-0.25, 0.35)$
Like B-S Delta, the modified Delta for \((a,b) = (-0.25,0.35)\) varies from \(-1\) to 0. Moreover modified Delta for \((a,b) = (\ln(0.85),\ln(1.15))\) is quite dissimilar to B-S Delta and is highly negative for ITM(in the money) options as it starts from below \(-1.4\) but converges to 0 like B-S Delta for OTM(out of the money) options (see Fig.4 (a)).

When the option being measured in deep ITM or OTM, Gamma is small. When the option is near or ATM Gamma is at its largest. In Fig.4(b), we observe that modified Gamma is almost alike to the B-S Gamma for \((a,b) = (-.25,.35)\). But for \((a,b) = (\ln(.85),\ln(1.15))\) , modified Gamma is largest near ATM but its value is very higher than B-S Gamma. Because within this small bounds, stock price is very volatile and when the stock price is very volatile there will be a high movement in the option price. Dissimilarity are very significant if we look on the ITM positions but for OTM positions, modified Gamma started to get very closer to the B-S Gamma. But for \((a,b) = (\ln(0.85),\ln(1.15))\), modified Gamma is largest near ATM but its value is very higher than B-S Gamma. Dissimilarity are very significant if we look on the ITM positions but for OTM positions, modified Gamma started to get very closer to the B-S Gamma. To speculate more specifically, we consider 3-D plot of Delta and Gamma for both models varying the stock price and time to maturity (see Fig.5 and Fig. 6). As the stock price increases along with time to maturity, put price decreases because we know that put become less expensive when stock price increases.

**Figure 5.** 3-D figures of Delta for both models
Table 2 and Table 3 shows that our derived Delta and Gamma coincides with B-S’s Delta and Gamma as we increase upper and lower bound.

**Table 2.** Delta of a non-dividend European put option for $S_0 = 49, K = 50, T = 0.3846, r = 0.05, \text{and } \sigma = 0.2$

<table>
<thead>
<tr>
<th>$(a, b)$</th>
<th>$[S_0 e^a, S_0 e^b]$</th>
<th>$\mu$</th>
<th>$\Delta_{MBS}$</th>
<th>$\Delta_{BS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\ln(0.85), \ln(1.15))$</td>
<td>[41.65, 56.35]</td>
<td>0.1523</td>
<td>−0.4513</td>
<td></td>
</tr>
<tr>
<td>$(-0.25, 0.35)$</td>
<td>[38.16, 69.53]</td>
<td>0.0201</td>
<td>−0.4856</td>
<td></td>
</tr>
<tr>
<td>$(-0.5, 0.5)$</td>
<td>[29.72, 80.79]</td>
<td>−0.0301</td>
<td>−0.4783</td>
<td>−0.4784</td>
</tr>
</tbody>
</table>

**Figure 6.** 3-D figures of Gamma for both models
Table 3. Gamma of a non-dividend European put option for \( S_0 = 49 \), \( K = 50 \), \( T = 0.3846 \), \( r = 0.05 \), and \( \sigma = 0.2 \)

<table>
<thead>
<tr>
<th>((a, b))</th>
<th>([S_0 e^a, S_0 e^b])</th>
<th>(\mu)</th>
<th>(\Gamma_p^{\text{MBS}})</th>
<th>(\Gamma_p^{\text{BS}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\ln(0.85), \ln(1.15)))</td>
<td>([41.65, 56.35])</td>
<td>0.1523</td>
<td>0.0887</td>
<td></td>
</tr>
<tr>
<td>((-0.25, 0.35))</td>
<td>([38.16, 69.53])</td>
<td>0.0201</td>
<td>0.0668</td>
<td></td>
</tr>
<tr>
<td>((-0.5, 0.5))</td>
<td>([29.72, 80.79])</td>
<td>-0.0301</td>
<td>\textbf{0.0655}</td>
<td>0.0655</td>
</tr>
<tr>
<td>((-0.8, 0.8))</td>
<td>([22.02, 109.05])</td>
<td>-0.0300</td>
<td>\textbf{0.0655}</td>
<td></td>
</tr>
<tr>
<td>((-2, 2))</td>
<td>([6.63, 362.06])</td>
<td>-0.0300</td>
<td>\textbf{0.0655}</td>
<td></td>
</tr>
</tbody>
</table>

5 Conclusion

The B-S and the modified B-S formulas for valuing European option prices assume that the market has no arbitrage opportunity and is complete. In our study, we derived the European put option price formula of the modified B-S model and explored its properties which clarify the soundness of this model. Through numerical experiments, we observed that the modified B-S price approaches the B-S price if the lower bound and upper bound become small \((a = -2)\) and large \((b = 2)\) enough. Effect of changes in stock price, strike price, time to expiration, risk-free rate, volatility on European put option are very similar for both models. Our derived formulas of Delta and Gamma for the modified B-S model coincide with the B-S model’s Delta and Gamma as we increase the range of the upper and lower bounds. This Delta and Gamma for the modified B-S model can play a significant role in financial markets to understand risk in different option position.

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